Week 10: Final Review!
MATH 4A

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Disclaimer: Since I am not the one writing the exam, I cannot guarantee this practice "exam" will look anything like the final. However, I reckon if you can do these without trouble, you're probably quite fine for the final.

4-1.5 Let $v = \begin{bmatrix} -4 \\ -6 \\ -8 \end{bmatrix}$, $u = \begin{bmatrix} -3 \\ -3 \\ 8+k \end{bmatrix}$, and $w = \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}$. The set $\{v, u, w\}$ is linearly independent unless $k =$?

Solution:

$\{v, u, w\}$ is linearly independent if the following condition is met: $c_1 v + c_2 u + c_3 w = \vec{0}$ if and only if $c_1 = c_2 = c_3 = 0$.

Note that $\{v, w\}$ (ie. without u) is linearly independent, since v is not a multiple of w. So, in order to make this set linearly *dependent*, we must find $c_1 v + c_2 w = u$. In other words, the following system must be consistent:

$$c_1 \begin{bmatrix} -4 \\ -6 \\ -8 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 8+k \end{bmatrix}$$

The augmented matrix corresponding to this system is

$$\left[\begin{array}{cc|c} -4 & -4 & -3 \\ -6 & -1 & -3 \\ -8 & 2 & 8+k \end{array}\right]$$

Reducing this into RREF, we get

$$\left[\begin{array}{cc|c} 1 & 0 & 3/4 \\ 0 & 1 & 3/10 \\ 0 & 0 & k+11 \end{array}\right].$$

The last equation corresponds to $k+11$, so $k = -11$ is what we need for this system to be consistent, in which case, $\{v, u, w\}$ linearly *dependent*. In other words, for $\{v, u, w\}$ to be linearly *independent*, we need $k \neq -11$.

4-2.5 Let $v_1 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Suppose $T(v_1) = \begin{bmatrix} -12 \\ 8 \end{bmatrix}$ and $T(v_2) = \begin{bmatrix} 19 \\ -9 \end{bmatrix}$. For an arbitrary vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$, find $T(v)$.

Solution: If we could find c_1 and c_2 such that $c_1v_1 + c_2v_2 = v$, then we would be done, since $T(v) = T(c_1v_1 + c_2v_2) = T(c_1v_1) + T(c_2v_2) = c_1T(v_1) + c_2T(v_2)$.

So, let's find c_1 and c_2 such that

$$c_1 \begin{bmatrix} -1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We note that this is a systems of equations, with the corresponding augmented matrix $\left[\begin{array}{cc|c} -1 & 1 & x \\ -2 & 3 & y \end{array}\right]$.

Row reducing this to RREF yields $\begin{bmatrix} 1 & 0 & -3x+y \\ 0 & 1 & -2x+y \end{bmatrix}$. This tells us $c_1 = -3x + y$ and $c_2 = -2x + y$.

Thus, we see $T(v) = c_1T(v_1) + c_2T(v_2) = (-3x + y) \begin{bmatrix} -12 \\ 8 \end{bmatrix} + (-2x + y) \begin{bmatrix} 19 \\ -9 \end{bmatrix} = \begin{bmatrix} -2x + 7y \\ -6x - y \end{bmatrix}$.

5-2.12 Let $A = \begin{bmatrix} -1 & -3 & -2 \\ 1 & 3 & 2 \\ -2 & -6 & -4 \end{bmatrix}$. Find a basis for the null space (kernel) of A.

Solution: This is the set of $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $Av = 0$.

We note that if $Av = \vec{0}$, then we have

$$\begin{bmatrix} -1 & -3 & -2 \\ 1 & 3 & 2 \\ -2 & -6 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x - 3y - 2z \\ x + 3y + 2z \\ -2x - 6y - 4z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We note that what we have above is a systems of equations, and we are trying to solve for x, y, z. The augmented matrix corresponding to this system is

$$\left[\begin{array}{ccc|c} -1 & -3 & -2 & 0 \\ 1 & 3 & 2 & 0 \\ -2 & -6 & -4 & 0 \end{array}\right]$$

which row reduces to

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This corresponds to $x + 3y + 2z = 0$, so if $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ was *any* solution, we must have $x = -3y - 2z$, so $v = \begin{bmatrix} -3y - 2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} z$. Since y and z were free variables, we see that they are "unconstrained" (ie. they can be any number). In other words, any solution would be of the form $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} z$, where y and z are scalars.

So, we see that $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis.

6-1.4 Find the determinant: $C = \begin{bmatrix} -1 & 2 & -2 & 0 \\ 0 & 0 & 3 & -1 \\ 3 & 0 & -1 & 0 \\ -2 & 1 & 0 & -2 \end{bmatrix}$

The solution to this problem is omitted, due to how annoying it would be to type up and the fact that this isn't very difficult to do.

7-1.10 Consider the ordered bases $B = (x, -(1+5x))$ and $C = (2, 2x-4)$ for polynomials of degree less than 2. Let $E = (1, x)$ be the standard basis.
Hint: Don't reinvent the wheel!

(a) Find T_C^E, the transition matrix from C to E.

(b) Find T_B^E.

(c) Find T_E^B.

(d) Find T_B^C.

Solutions: First, we write $B = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \end{bmatrix} \right\}$, and $C = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right\}$.

Now...

(a) $T_C^E = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix}$

(b) $T_B^E = \begin{bmatrix} 0 & -1 \\ 1 & -5 \end{bmatrix}$

(c) $T_E^B = \begin{bmatrix} 0 & -1 \\ 1 & -5 \end{bmatrix}^{-1}$

(d) $T_B^C = T_E^C T_B^E = (T_C^E)^{-1} T_B^E = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & -5 \end{bmatrix}$.

8-1.8 Consider $A = \begin{bmatrix} 7 & 5 & -6 \\ -6 & -4 & 6 \\ 5 & 5 & -4 \end{bmatrix}$. Find the eigenvalues of A and its corresponding eigenvectors.

Solution: It can easily be seen that the characteristic polynomial is $-\lambda^3 - \lambda^2 + 10\lambda - 8$, which has roots $-4, 1, 2$ (ie. these are our eigenvalues).

Take $\lambda = 1$. We note that $A - \lambda I = A - I = \begin{bmatrix} 6 & 5 & -6 \\ -6 & -5 & 6 \\ 5 & 5 & -5 \end{bmatrix}$.

We notice that $A - I$ can be row reduced to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which tells us the null space of $A - I$ has elements of the form $\begin{bmatrix} s \\ 0 \\ -s \end{bmatrix}$, which is generated by $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Any of these (except the 0 vector) is an eigenvector associated with $\lambda = 1$.

The eigenvectors associated to the other eigenvalues can be found similarly.

9-1.1 Let $A = \begin{bmatrix} 6 & -3 & -13 \\ 1 & 2 & 5 \\ 3 & -3 & -10 \end{bmatrix}$. Suppose $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are eigenvectors. Then what are the eigenvalues?

Solution: First, let $v_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

We note that $Av_1 = \begin{bmatrix} -6 + (-3) + (-1)(-13) \\ -1 + 2 - 5 \\ (-1)(3) + 1(-3) + (-1)(-10) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = -4v_1$, in which case, we see that v_1 is an associated eigenvector to -4. We can find the other eigenvalues similarly.

9-1.4 Let $A = \begin{bmatrix} 5 & 2 \\ 0 & 3 \end{bmatrix}$. Diagonalize A. Compute A^{500}.

Solutions: It can easily be checked that the characteristic polynomial of A is $(\lambda - 5)(\lambda - 3)$, which has roots 5 and 3, which are our eigenvalues. So, one candidate for D would be $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

Looking at 5, we see that $A - 5I = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}$. It can easily be checked that the null space of $A - 5I$ is $\left\{ \begin{bmatrix} s \\ 0 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$, which has basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. We similarly see that $A - 3I = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$, which has kernel $\left\{ \begin{bmatrix} -s \\ s \end{bmatrix} \middle| s \in \mathbb{R} \right\}$, which has basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. So, we can construct $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Given this, P^{-1} can be found rather easily.

Now, ask yourself: Why is it now "easy" to find A^{500}?

9-1.11 Let $A = \begin{bmatrix} -4 & 0 & 0 \\ -1 & -5 & 1 \\ -3 & -1 & -3 \end{bmatrix}$. Find the real eigenvalue of A, it's multiplicity, and the dimension of its eigenspace.

Solution: It can be readily checked that the characteristic polynomial of A is $-(\lambda+4)^3$, in which case, the only eigenvalue is -4.

We see that $A - (-4)I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 1 \\ -3 & -1 & 1 \end{bmatrix}$. To find the dimension of the eigenspace of -4, we must find the dimension of the nullspace of $A + 4I$. We note that A can be row reduced to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. From this, we see that the solution is of the form $\begin{bmatrix} 0 \\ s \\ s \end{bmatrix}$ for $s \in \mathbb{R}$, which tells us that the null space has basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ (ie. it has dimension 1).